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A FACTOR ANALYTIC MODEL FOR LONGITUDINAL DATA

by

CLEMENT DASSA

Department of Educational Theory

A Thesis submitted in conformity with the requirements
for the Degree of Doctor of Philosophy in the
University of Toronto.

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UNIVERSITY OF TORONTO

SCHOOL OF GRADUATE STUDIES

PROGRAM OF THE FINAL ORAL EXAMINATION
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

OF

CLEMENT DASSA

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Room 309, 63 St. George Street

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ABSTRACT

A factor analytic model for longitudinal data in the social sciences is presented. The model is a fixed restricted factor analysis model that allows for the simultaneous estimation of factor loadings and factor scores. The factor loadings are restricted to remain invariant over occasions, and the factor scores are restricted to three laws of development. The first law represents an unconstrained time-course, the second a constrained linear time-course, and the third a constrained nonlinear time-course.

Two procedures for estimation of the parameters are used: a least squares method and a likelihood ratio method. The corresponding loss functions are minimized by a Conjugate Direction algorithm. Related problems such as the rescaling of the parameters and the determination of proper starting points are presented and solutions proposed.

A simulation procedure is presented that generates data under controlled conditions. Both simulated and real data are used to illustrate the theory and estimation procedures developed in the study. The necessary computer programs were written in FORTRAN IV and used to obtain numerical results which yield information about the applicability of the model to data analysis in the social and behavioral sciences.

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CHAPTER 1

GENERAL INTRODUCTION

Longitudinal data in the social sciences may be defined as a set of repeated measures of the same observed units, often individuals, over a period of time. It allows for the measurement of a trend relative to a trait or family of traits with respect to an individual or a group. Longitudinal studies are primarily concerned with the "development" or progress of an entity measured over a period of time. In opposition to cross-sectional studies which refer mainly to a static view of the field investigated, longitudinal studies "are socio or psycho-dynamic and are interested in change" [Wall and Williams, 1970, p. 7].

A variety of models and techniques exist that allow the researcher to investigate particular aspects of change. When the emphasis is on exploring the latent structure of the data then factor analytic models can be used. In general, information on both the overall structure of the variables studied and on individual profile or trend is needed. This is highlighted by the recent emphasis in education on individual achievement and development through individualized instructional programs. In developmental psychology such information is essential; in other branches of the social sciences it is often useful.

To date, all the factor analytic models and techniques used to analyse longitudinal data are based on the classical "random" approach,

i.e., the factor scores are assumed to be random variables. Recently, McDonald (1974c, ~~1974d~~, in press) has proposed a fixed factor model where the factor loadings and the factor scores are fixed parameters to be estimated. As a result, the simultaneous estimation of factor loadings and factor scores seems to open new possibilities for the analysis of data in an experimental context (McDonald, 1974d).

The objectives of this study may be described briefly as follows: to apply McDonald's approach to longitudinal data by specifying plausible curves for "mental growth" and by restricting the factor patterns within as well as across occasions; to develop an adequate procedure for the estimation of parameters and hypothesis testing; to establish the feasibility and applicability of the proposed model by numerical illustrations; and, last, to explore different related problems.

In a sense, the approach underlying the model could be viewed as a conjunction of three main approaches: fixed factor analysis, restricted (confirmatory) factor analysis, and growth curves studies.

In the sequel, we give a review of methods used to analyse longitudinal data in Chapter 2, together with a discussion of the proposed strategy of analysis. In Chapter 3, we present the model with its three specifications of the factor score trends: unconstrained linear and non-linear; the least squares and maximum likelihood methods of estimating the parameters are also presented in this chapter.

Next, we illustrate the model. In the first part of Chapter 4, we present the following related topics: the simulation procedure, the patterning and packing of the parameters, the minimization algorithm, the rescaling of the parameters, and finally, the determination of adequate starting points. Numerical illustrations with both simulated and real data are presented in the second part of Chapter 4.

Finally, in Chapter 5, a summary and discussion as well as some conclusions are given.

CHAPTER 2

A LITERATURE REVIEW2.1. Introduction

The measurement of change is a long standing problem which has generated abundant literature spanning many branches of the social sciences. Besides psychology, education is the domain which is perhaps the most concerned with this problem. From the foundations of education — seeking to understand the basic cognitive processes — to the evaluation of instructional programs and of educational systems, the study of change remains a central preoccupation of researchers.

The literature is very often unclear about the exact meaning of change. As a result — and also because of the intrinsic nature of the problem — certain procedures and models used to measure change present serious shortcomings. A convenient way to categorize the different strategies of analysis depends on the central interest of the research. We can number two families of such interest. First there are studies centering on the variation of the mean score of a group of students. The problems encountered in this approach belong to the field of analysis of variance of repeated measures. Second there are studies in individuals of variations in development where the focus is on "pattern" or "profile" of growth. In this latter approach the problems can further be of two basically different types: the emphasis can be on the prediction of an

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individual's performance relating to a well defined characteristic or ability or, it can be on the laws of growth or genetic change, the object being to define these laws.

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In this chapter, we will review in Section 2.2. the univariate and multivariate models of analysis of variance for repeated measurement. In Section 2.3 we will review the different procedures and models, other than the factor analytic ones, used to analyse longitudinal data; this section will include a discussion on the gain (or difference) score as well as various regression models used to predict change. In Section 2.4 we will review the various factor analytic approaches as well as the specific factor analytic models that have been proposed for longitudinal data. Finally, in Section 2.5 we will introduce the basic concepts underlying our model.

2.2. Analysis of group trends for longitudinal data

In this section we will discuss univariate and multivariate analysis of variance for behavioral experiments concerned with the trend of group means. The subjects forming the groups are repeatedly measured with respect to a well defined trait. Whenever a univariate analysis is appropriate the longitudinal or repeated measures design is the same as the randomized block design in the case of one sample of subjects and the so called split-plot design in the case of multiple samples. The design used could be truly experimental or quasi-experimental. Campbell and Stanley (1963) have extensively discussed the conditions under which a design might meaningfully be employed by behavioral researchers when thorough experimental controls are impossible. We will therefore consider the case of experimental designs.

Longitudinal data can be classified with reference to one or more experimental factors*. One factor that is always present is the occasion effect. In the simplest case this factor is fixed and has two levels corresponding to a pretest and a posttest with one experimental treatment. In the more general case there may be more than two levels with a corresponding sequence of treatments. This design on the occasions could be combined with other factors to define an experimental design. In general, the subjects are considered as a random factor. In the sim-

* In this section, the term "factor" has its usual analysis of variance meaning.

plest case, the unique level of this factor is determined by a single sample, whereas, more generally, there may be multiple groups. The groups could represent different subpopulations or could be subjected to different treatments. Whatever the sampling design, the subjects within groups are always crossed by occasions.

In these longitudinal studies as in all experimental studies the main purpose is to determine whether the experimental intervention changes the behaviour under investigation. This is achieved through a plot of the means vs. time. In the case of a single group experiment, we wish to determine the shape of the time-curve. In the case of a multiple group experiment — be it a randomized experiment where the groups are subjected to different treatments or a comparative study where they represent different segments of the population — the main objective is to compare the mean time-curves of the groups.

A comment on the nature of longitudinal data is in order here. Though different measures are obtained from each subject over time, it should be noted that the repeated measures originate from the same test or parallel forms of the test — that is, from a single variable. However, such data are, strictly speaking, multivariate in the sense that multiple measurements are obtained for each subject. In general, we must assume that the population covariance matrix C of the repeated measures in the population of subjects has a general (unrestricted) pattern since the repeated responses of each subject must be assumed to have correlated sampling variation.

Under more restrictive assumptions on C , univariate procedures are readily available. A detailed account of these procedures is given by Gaito and Wiley (1963). They considered the two following cases where the data conform to the mixed model assumptions:

$$(1) \quad C = -\sigma^2 I,$$

i.e., C has equal variance and zero covariance, and

$$(2) \quad C = \sigma_a^2 \mathbf{1}\mathbf{1}' + \sigma^2 I,$$

i.e., C has equal variance and constant covariance. Following earlier accounts of trend analysis (Lindquist, 1947 and 1953; Edwards, 1960; Winer, 1962) they used the orthogonal polynomial approach. In this approach the sum of squares of the effects is divided into orthogonal components associated with functional forms (linear, quadratic, etc.). Typically, the different levels on the occasion factor are regarded as steps along an underlying continuum (time). There exist different methods for exploring these continua and for specifically assessing the trend over occasions; Cochran and Cox (1957) have given a survey of the most important ones. For equally spaced occasions, the orthogonal polynomial coefficients are tabled in Fisher and Yates (1957), otherwise the coefficients are obtained by solving a series of simultaneous equations (Wishart and Metakides, 1953). A simple method is provided by Robson (1959) and a numerical construction of the coefficients using a recurrence formula is given by Emerson (1968).

Although the assumption of equal variance is not very restrictive, because of the well known robustness of the F test (see for example Scheffé, 1959, and Boneau 1960), the assumption of zero covariance proves to be inadequate; in most studies it is more plausible to assume that the sampling errors are correlated. Hence, assuming that the correlation of all pairs of levels of the occasion factor across the population of subjects is the same, that is: $C = \sigma_{\alpha}^2 \rho + \sigma^2 I$, is more realistic than assuming that $C = \sigma^2 I$. The statistical procedures are similar in the two cases; except that the expected mean sum of squares in the correlated case depends on the constant correlation. Various accounts of such procedures are given by Winer (1962, 1971), Gaito and Wiley (1963) and Bock (1963, 1975). In the event that the correlation is not constant, the F test relative to the occasion factor is not valid: the sampling distribution of the F statistic has degrees of freedom smaller than the degrees of freedom under the model (Box, 1954). On the basis of further investigation of the effect of heterogeneous correlations by Greenhouse and Geisser (1959) and Lana and Lubin (1963), Glass and Stanley (1970) suggest a contingent hypothesis testing procedure resorting to the multivariate technique of Hotelling's T^2 test. Gaito and Wiley (1963) discussed the extent to which univariate procedures are feasible in this case.

The univariate procedure clearly cannot be used to treat the general case. We have to resort to a multivariate approach of analysis of variance (Anderson, 1958). Bock (1963, 1975) has provided a general multivariate treatment of the repeated measures design. The univariate

cases (1) and (2) previously formulated become special cases of a general treatment. Not only does the multivariate formulation prove to be conceptually more appealing but it presents computational advantages over the univariate treatment as well. We will now briefly outline this procedure.

In the case of p occasions and n groups it is assumed that the observation for the i -th subject on the j -th group is Y_{ij} a p -variate vector:

$$Y_{ij} = \underline{\tau} + \underline{\theta}_j + \underline{\varepsilon}_{ij},$$

where $\underline{\tau}$, $(p \times 1)$, is a vector mean for occasions, $\underline{\theta}_j$, $(p \times 1)$, is a vector effect for the population, and $\underline{\varepsilon}_{ij}$, $(p \times 1)$, is a vector of sampling errors assumed to be distributed as $N(0, C)$. It suffices to consider only differences in the population mean time-curves parametrized by $[\underline{\theta}_j]$. If the occasions are equally spaced then the design-model set-up can be conveniently retained and reparametrized in orthogonal polynomial contrasts. The polynomial representation is then given by:

$$\underline{\theta}_j = \beta_{0j} \underline{1} + \beta_{1j} \underline{x} + \beta_{2j} \underline{x}^2 + \dots + \beta_{kj} \underline{x}^k, \quad k \leq p-1,$$

where \underline{x} , $(p \times 1)$, is the vector of fixed time points and $\underline{x}^j = [x_i^j]$, with $i=1, \dots, p$ and $j=1, \dots, k$. Denoting $\underline{\theta}_j$ in matrix notation we obtain

$$\underline{\theta}_j = X \underline{\beta}_j,$$

where $X = [\underline{1} \ \underline{x} \ \dots \ \underline{x}^k]$ and $\underline{\beta}_j = [\beta_{0j} \ \beta_{1j} \ \dots \ \beta_{kj}]'$. After reparametrization we have

$$\underline{\theta}_j = P \underline{Y}_j,$$

where P is the matrix of orthogonal polynomials and the components of \underline{Y}_j are the orthogonal polynomial coefficients. Conceptually, the original observations are transformed so that

$$P'Y_{ij} = \underline{\tau}^* + Y_j + \underline{\epsilon}_{ij}^*$$

where $\underline{\epsilon}_{ij}^* \sim N(0, P'CP)$. Practically, only the sum of square and product (SSP) matrices are transformed by P . The transformed SSP matrices allow for a stepwise polynomial regression analysis. Matching the different structural assumptions on C there are different ways of using the transformed SSP matrices to test the polynomial of least degree capable of representing $\underline{\tau}$. Three assumptions are discussed by Bock (1963, 1975). The first one is that of equal variance and equal covariance. In this case $P'CP$ is a diagonal matrix whose last $p - 1$ elements are homogeneous. A second less restrictive assumption is that $P'CP$ is diagonal but its last $p - 1$ elements are not necessarily homogeneous. Finally, $P'CP$ is assumed to be a general covariance matrix, i.e., the error components of $\underline{\epsilon}^*$ are correlated as well as heteroscedastic. All these assumptions can be tested in large samples using a likelihood ratio test (Jöreskog, 1970; McDonald, 1974a). Under the first and second assumptions we can test each polynomial component separately using a decision theoretic approach thus determining the degree polynomial that best describes the data (Anderson, 1962). Under the more general assumption, the tests of the polynomial effects must be multivariate.

It should be noted that the use of the polynomial analysis is based on the ordered metric properties of the occasion factor and is applicable when the occasions correspond to points on a continuum. However, in certain design structures where the occasions correspond to a crossed and/or nested classification of treatments such as hierarchically ordered treatments and 2^n factorial combinations of treatments, the parameters of the model are appropriately characterized by treatment contrasts and interactions (Bock, 1975).

So far, we have only discussed the case where one variable was repeatedly measured over time. Typically, in longitudinal studies in the social sciences, a number of qualitatively distinct variables are measured. The multivariate analysis of the "multiple-variable" case proceeds from an extension of the analysis discussed in this section; appropriate multivariate test statistics as well as approximate tests are then used (see Huynh, 1978, for a recent review and contribution). The only serious problem which arises under the general assumption is the choice of grouping of the variables to be tested jointly or their ordering for purpose of step-down tests. These choices are left to the researcher and are based on considerations of substance.

Whether the analysis is univariate or multivariate, or whether the polynomial analysis is used or not, the analysis focuses mainly on the general trend of the group or groups investigated based on averaging individual responses. Its main interest lies in assessing the evidence for an occasion effect and to characterize the effect if it is present.

Although individual growth curves of the variables measured can be examined, no investigation of the latent structure of the variables used in the general multiple-variable case is available through analysis of variance. Also absent is an individual score giving information on each subject's particular profile or trend. Such individual-oriented approaches are presented in Section 2.3.

2.3. Gain score as an individual and group measure of change

The simplest score pertaining to an individual measure of change is the gain score defined as the difference between a premeasure and a post-measure. Such a score has been widely used in the past and is still used even though it involves very serious conceptual problems. These problems have been extensively discussed in the literature and as a result, more appropriate approaches to the measurement of individual change are now available.

The fundamental problem of the difference score resides in the error of measurement. Whereas mean differences of observed scores are equal to mean differences of true scores, individual scores are affected by the error of measurement. It follows that the correlation between an initial observed score x and an observed gain score g is biased (Thorndike, 1924). Thomson (1924, 1925) and Zieve (1940) attempted to eliminate this bias by adjusting the correlation formula of the observed scores. The usual attenuation is enhanced in this case. Thorndike (1966)

used the supplementary information provided by parallel premeasures to correct this bias. Different ways of correcting for attenuation could arise if regression estimates of true scores are used (Wiseman and Wrigley, 1953; Lord, 1956, McNemar, 1958); this in turn, could cause a problem of choice among the different available formulae.

Since the initial-gain correlation does not yield a clear picture of the relation between initial true score X and true gain score G , defined as $G = Y - X$ (where Y is the final true score), another approach is thus necessary. Instead of asking what is the relation between X and the part of Y "different" from X (i.e., $Y - X$), let us ask how we can gain information on Y using X , in other words, how we can predict Y from X .

Following Garside's (1956) and Thorndike's (1966) arguments, O'Connor (1972a) clearly established that $\rho_{XG} \geq 0$ if and only if $\beta_{Y.X} \geq 1$, where ρ_{XG} is the initial-gain correlation and $\beta_{Y.X}$ is the regression coefficient in the prediction of Y from X . Hence, $\beta_{Y.X}$ gives the same information as ρ_{XG} . However, when ρ_{XG} is interpreted in the usual way of positive or negative value meaning positive or negative relations, it proves to be "artificial and misleading" (O'Connor, 1972a). It is therefore preferable to express the relationship between G and X in terms of the relationship between Y and X .

In the same way, it is preferable to express the relationship between the gain G and some other variable W in terms of an equivalent.

relationship between X , Y and W . Lord (1963) argued that the proper correlation that should be considered in such a relationship is the partial correlation $\rho_{GW \cdot X}$ which is clearly equal to $\rho_{YW \cdot X}$. Shifting from the correlational to the regression point of view, Werts and Linn (1970) showed that $\beta_{GW \cdot X} = \beta_{YW \cdot X}$.

In an effort to free the gain from the initial status a residual gain score obtained by partialling out X from Y has been defined (Du Bois, 1957; Manning and Du Bois, 1958, 1962). The true residual gain γ is defined by: $\gamma = Y - \beta_{Y \cdot X} X$. Basically γ is a difference between the final score and that part of the final score that can linearly be predicted from the initial score. As Cronbach and Furby (1970, p. 74) put it:

"One cannot argue that the residualized [gain] score is a "corrected" measure of gain, since in most studies the portion discarded includes some genuine and important change in the person".

However, such a score could be useful in identifying individuals whose change has been exceptionally high or low and in comparing groups (Dyer et al, 1967, 1969).

Using the same line of reasoning, Tucker et al (1966) proposed a "base-free" measure of change "primarily intended for correlational work". They provided a biased estimator of γ which does not give the least-squares estimate of individual base-free scores. Hence, no satis-

factory interpretation of these scores is available through this "base-free" approach. The problems caused by such an estimator are discussed by Cronbach and Furby (1970) and by O'Connor (1972b) who provide proper least-squares estimators of the true residual gain γ .

The problem of proper estimation of gain scores in the presence of error of measurement is linked to the well known unreliability of the observed score g . In the classical test theory, there exist different definitions of reliability. Although they are equivalent for single observed scores, they are not equivalent in the case of gain scores which are composite scores. Lord (1963) and Stanley (1967) have provided general and special formulas for the reliability of change scores. Traub (1967, 1968) and Glass (1968) discussed the problem of the reliability of residual scores. Hoffman (1963) has studied the reliability of repeated measures when practice effects are present. Webster and Bereiter (1963) and O'Connor (1972a) concluded that parallel-forms reliability is the best estimate of the reliability of gain scores.

The unreliability of the observed gain score makes it a poor estimator of the true gain G . Other estimators are thus important. Lord (1956) proposed a least-squares estimator \hat{G} of the true individual gain based on a multiple regression of G on x and y :

$$\hat{G} = \bar{g} + b_{Gx \cdot y}(x - \bar{x}) + b_{Gy \cdot x}(y - \bar{y})$$

McNemar (1958) arrived at the same result using less restrictive assump-

tions than Lord's. Further extensions and clarifications of Lord's paper are given by McNemar (1958), Lord (1958, 1963) and Davis (1961). The solution proposed gives a very practical way of estimating true individual gain through a partition of the pretest - posttest scatterplot into classes of observations having approximately equal gains. This method is based on the implicit assumption that the initial and final observations are based on the same test or on parallel forms. Item analysis procedures which are optimal with respect to the detection of change and thus provide adequate parallel forms for analysing change have been proposed by Gruber and Weitman (1962), Bereiter (1963) and Saupe (1966). Cronbach and Furby (1970) and O'Connor (1972a) have extended the Lord-McNemar multiple regression approach by using additional variables which allow for more precise estimates of change scores.

Let us now turn to the problem of using gain scores in the comparison of group change due to treatment effects. Basically, two distinct techniques are available. We can analyse the gains using an analysis of variance approach discussed in Section 2.2, or we can use an analysis of covariance (Stanley, 1966). In general, these techniques do not yield the same results (Lord, 1967). Engelhart (1967) discussed the special case where the two groups are paired on the pretest. In this case the F tests used in both analyses give the same results. Bock (1975) has shown that the gain analysis (the analysis of variance of gains) is based on an unconditional distribution of gain scores, whereas the analysis of covariance is based on a conditional distribution of gain on initial score. The two

approaches provide answers to different questions. In the analysis of gains the researcher asks if there is a difference in the average gains of the populations from which the groups are drawn. In contrast, in the analysis of covariance the researcher is in fact asking if a subject belonging to a group is expected to show more gain than a subject of another group given that they have the same initial score. The choice between the two methods depends upon the sampling procedure and the inferential problem. If the subjects are randomly drawn from defined populations and if the objective of the analysis is to compare the average gain of these populations, then both analyses are adequate. In the case where the subjects are assigned randomly to treatment groups, analysis of covariance is usually relatively more efficient and should preferably be used. It should be noted that both analyses are equally efficient in the special case where the correlation between the initial scores and the final scores is equal to one. In the case of a biased assignment of subjects to treatment groups, the analysis of covariance is the only appropriate method. However, it should not be overlooked that, in both cases, the absence of true randomization could lead to spurious results (Campbell and Stanley, 1963; Campbell and Erlebacher, 1970; Humphreys, 1976).

This section demonstrates that the formulation of most problems centering on individual or group change in terms of gain scores is generally inadequate because it conceals intrinsic conceptual difficulties that can lead to an erroneous interpretation of results. Multiple regression and analysis of covariance respectively used for individual and group assessment of change can always provide clearer answers provided that

researchers frame their questions appropriately (Cronbach and Furby, 1970; Linn and Slinde, 1977).

2.4. Factor analytic approaches and models for longitudinal data

The analysis of variance approach presented in Section 2.2 proves very useful for determining the mean change of a group of subjects. The gain score approach seen in Section 2.3, though presenting serious conceptual difficulties, could be of some use in certain cases. However, none of the previous approaches allow reducing the set of observed variables to underlying latent variables and studying "patterned" change. Factor analysis offers a useful approach in this respect. Typically, the researcher trying to investigate the change that occurred in longitudinal studies will address the problem of identification and invariance of the factor structure over time as well as describing change in individual profiles on the dimensions defined by the factor pattern.

A number of techniques exist that analyse the correlation or covariance matrix among all variables for each occasion separately. In this case, the main problem is whether the different sets of factors obtained are similar among themselves. In principle, congruence or hypothesis rotation could be used to solve this problem. Bentler (1973) has discussed the shortcomings of these solutions. The main difficulty with separate analyses is the uncertainty over the similitude of the different factors since the rules defining them are not the same. We will therefore turn to the different models that simultaneously analyse the data for all occasions.

Within the framework of factor analysis, the concept of change can be investigated through the stability and change of the factor loadings and scores. We can distinguish three main approaches among the different attempts to factor analyse change:

- (1) keeping the factor scores invariant and allowing the factor loadings to change,
- (2) keeping the factor loadings invariant and allowing the factor scores to change,
- (3) allowing both factor scores and factor loadings to change.

Harris (1963) and Evans (1967) have provided models of the first type.

Harris (1963) used a canonical factor model for the description of change over only two occasions. In this approach the variables on the two occasions were rotated simultaneously; this procedure amounts to keeping the factor scores constant and allowing the correlations of the variables with these factor scores (the factor loadings) to vary. This is equivalent to stating "that the individuals have not changed, but the variables may have". It is therefore possible to obtain more factors than variables. According to Harris, his procedure can be used to approximate the factor structure of the covariance matrix of the gain scores. The examples he provided seem to confirm the main shortcoming of the gain scores

already discussed in Section 2.3, namely that such a procedure does not prove very informative.

Another model which postulates constant scores and varying factors has been proposed by Evans (1967). This model also allows for the analysis of growth in mean scores. A supplementary factor on which the individuals being studied do not vary is postulated. A statistical test determines if such a factor is required by the data. Strictly speaking this amounts to allowing the factor scores to change on the average only. Two sets of constants are built into the model to take into account the arbitrariness of the origins and units of the tests; their function is to standardize the variance of the tests and to render comparable the units of measurement for each test. The model provides least-squares estimates of the mean and the individual factor scores. An analytical method of rotation is provided solely in the special case where the growth functions defined by the loadings are proportional. This is in fact an application of Cattell's parallel proportional profile principle (Cattell, 1955).

It should be noted that keeping the factor scores identical causes the correlation between the same tests across occasions to be accounted for by the common factors. This means that variance that is specific to each variable is considered as common variance and is included in the analysis. This is equivalent to keeping the unique scores for variables on the first occasion uncorrelated with the corresponding unique scores

on subsequent occasions (Bentler, 1972). Cattell (1963) and Jöreskog (1969) have provided such models. In order to remove the specific variance from the analysis, one must allow the unique factors to be related (McDonald, 1969; Bentler, 1972).

Cattell (1963) has elaborated on two specific applications of factor analysis to the measurement of change that he introduced earlier (Cattell, 1946, 1952). His P-technique consists of repeatedly measuring one person on a set of variables over a period of time, with the series correlated and factored. In his incremental R-technique a group of subjects are measured on a set of variables over two occasions; the factor analysis is carried on the correlation matrix of the difference scores. P-technique deals with "person-unique" factors instead of the common factors of incremental R-technique. The use of difference scores in incremental R-technique raises the familiar problems of reduced reliability, correction for initial measures and the usual regression to the mean on the final measurement; all of these problems have been discussed in Section 2.3. Another important restriction of this technique is that it takes into account only two occasions. As for the P-technique it also has its share of problems. For one thing, we can expect its longitudinal reliabilities to be systematically lower than those in the R-technique. But even more important, the sampling of its occasions in time and the related problem of degrees of freedom for significance testing need more investigation. The effect of special time-related influences in

the P-technique is discussed by Cattell (1963). Instead of partialling out time before factoring, it is recommended to investigate if the factors obtained from the original correlation matrix tend to have significant relations with time. Time could sometimes appear as a simple structure second-order factor. A procedure consisting of trying different time intervals of each variable on a factor in order to maximize their correlation is presented by Cattell (1963).

The problems involved in the P-technique are basically the same as those encountered in time-series analysis. Recently, as the need for proper statistical treatment of the single case has grown, this type of approach has been used increasingly in psychology and education (Holtzman, 1963; Anderson, 1963; Gottman *et al.*, 1969; Nishisato, 1971; Frederiksen, 1974). However, its use is hampered by practical difficulties such as practice and boredom effects. Even though solutions to these problems exist in principle, the fact that a large number of observations is needed imposes very serious limitations.

One further limitation of the P-technique that is of special importance here, stems from the assumption that the unique scores are uncorrelated over the occasions. Cattell (1963) defends this assumption on the basis of a general recommendation: namely, that the part of error of measurement that is most detrimental to the analysis should be considered as an exogenous factor and studied separately. In response, Anderson (1963) argued convincingly in favor of correlated unique scores.

Jöreskog (1969) presented a somewhat more general approach applicable to a group of subjects on more than two occasions. Again the unique scores are assumed to be uncorrelated over time.

The models where the factor scores are invariant and the factor loadings are varying are the most common but their usefulness is restricted to special situations where the factor patterns differ over time while the ordering of subjects along a general continuum remains invariant. In general, the assumption of perfect stability of the factor scores proves to be unrealistic. The main objections to these models were discussed by Corballis (1965), Corballis and Traub (1970), Baltes and Nesselroade (1973), Bentler (1973) and Corballis (1973). They can be summarized as follows:

(1) Since test scores will usually change, it seems unreasonable not to allow factor scores to change.

(2) By essentially forcing the factor loadings to change, the interpretation is made difficult.

(3) Too many factors are obtained for "economical description".

An alternative approach consists in the converse of the previous one. The factor scores are allowed to change while the factor loadings are

kept constant (or constrained to be as alike as possible). This procedure still tends to confuse the issue of interpretation. The situation in which factor scores change on a given factor can be interpreted as one in which different factors (though possibly correlated) are measured on different occasions.

A model using this approach has been derived by Tucker (1963) as an application of his three-mode factor model to longitudinal data. In this model, the individuals, the traits and the occasions define three distinct modes labeled "observational modes" and "intrinsic modes" depending on whether they are related to observed measures or idealized entities. Basically, two different ways of applying three-mode factor analysis to longitudinal data are possible: keeping the relations of observed traits to intrinsic traits (the factor loadings) constant over occasions and assigning all changes over occasions to changes in the individual's factor scores, or keeping the factor scores constant while the factor loadings are allowed to change. It should be noted that the factor scores and the factor loadings cannot vary simultaneously. In his application to longitudinal data Tucker used the first way. The resulting model is truly an extension of principal component analysis rather than a factor analysis model, in the sense that the generalization of the communality problem remains unsolved. The choice of transformations of the intrinsic modes constitutes another major unsolved problem.

The third and final approach (allowing both factor scores and factor loadings to vary) was first presented by Corballis and Traub (1970).

In their model it is assumed that the same factors underlie the tests on each occasion but both factor scores and loadings are allowed to change. Although factor loadings are allowed to change, it is hoped that they will remain constant. When the factor loadings are not constant it creates a somewhat difficult problem of interpretation. Though the model generally provides a unique orthogonal rotation solution, this same desired feature creates the following paradoxical problem: if a given solution is not readily interpretable it cannot be rotated to yield a solution which can be interpreted more easily. This stems from the assumption of orthogonality within as well as between occasions. Hence, one possible way to improve the model would be to consider oblique solutions. However, oblique solutions could not compensate for the basic limitations of all the previous models. This limitation is the lack of a direct solution for the factor scores. No measures of individual changes are obtained, instead, correlational indices of factor score change are typically obtained. In this respect, it should be noted that the Corballis-Traub solution maximizes temporal stability of factor scores and thus may not prove adequate for certain types of change usually encountered in short-term situations (e.g., mood and state) where change is unstable, i.e., the cross-occasion correlations are low and occasionally negative (Nesselroade, 1972). The model has two further limitations: it does not apply to more than two occasions and it uses a double stage least-squares estimation procedure. Corballis (1973) has extended the model to more than two occasions. He pointed out that least-squares and maximum-likelihood solutions could have been obtained through the use of ACOVS,

a computer program devised by Jöreskog et al (1970) had it not been for severe storage requirements; he thus retained a double stage least-squares estimation.

Swaminathan (1978) extended Corballis' approach. In this model, the assumptions are basically the same as Corballis' except that the factor scores follow a first order auto-regressive rule. It then follows that the interoccasion matrix (for different occasions) of the factor scores correlations is given by

$$r(x_j, x_i) = D_{j+1} D_{j+2} \dots D_i, \quad j = 1, \dots, i-1,$$

where x_i is the vector of factor scores on occasion i and D_i is a diagonal matrix of constants, whereas, in Corballis' model

$$r(x_j, x_i) = D_{ij}, \quad j \neq i,$$

where D_{ij} is a diagonal matrix of correlations between the same factors across different occasions. The model provides maximum likelihood estimation of the parameters. This allows in turn for statistical tests of joint hypotheses about the parameters as well as structural hypotheses of interest such as factorial invariance. Since no illustration of the model is yet provided it is difficult to assess its practicality. Let us however underline the fact that no solution is provided for individual factor scores. As in Corballis' model, measures of individual changes are obtained through correlational indices of factor scores between occasions.

2.5. A different strategy: Fixed and restricted factor analysis

Because the emphasis in longitudinal studies is on development or growth, the main concern of specifically designed factor analytic models should be placed on exploring developmental or growth functions. In life-span longitudinal studies these functions could be defined as "the form or mode of the relationship between an individual's age and the changes occurring in his responses on some specified dimension of behavior over the course of his life" [Wohlwill, 1970, p. 151]. In other areas of importance in education — such as instructional learning — growth functions can be defined in a similar way except that the time span is generally shorter. In addition to centering its inquiries on intraindividual variation (i.e., differences within persons), longitudinal methodology focuses on interindividual variation (i.e., differences among persons) in intraindividual change (see, e.g., Nesselroade and Reese, 1973).

In the framework of factor analysis, factor scores are individual measures. If we take the point of view that factors may be considered as potential tests then factor scores in longitudinal models ought to be allowed to vary since test scores usually do vary over time. Thus, growth curves could be expressed in terms of changing factor scores. As for the factor patterns, it is evident from Section 2.4 that they should be allowed to change. However, a close similarity of the factors is desirable for the sake of interpretation. Situations where the factor patterns vary could either denote error factors or "situation-specific" factors [Baltes and Nesselroade, 1973, p. 225]. Hence, an adequate

approach to the problem of designing a longitudinal model seems to be to allow the factor patterns to change and to specify certain laws according to which the factor scores could change. This approach raises the problem of the indeterminacy of factor scores and the problem of specifying adequate laws according to which the factor scores would change.

First, let us examine the nature of the indeterminacy of factor scores and the ways of reducing if not eliminating it. All the models presented in the previous section are classical unrestricted common factor models where the factor scores are random variables. A well known limitation of these models is the indeterminacy of factor scores arising from the inability to determine more factors than the existing observed variables. Guttman (1955) demonstrated that any factor score matrix can be expressed in terms of an arbitrary matrix by using a Kestelman (1952) construction. McDonald (1974b) re-examined the problem of factor indeterminacy, reinterpreted its meaning and reassessed its extent. A review of the topic is given by Steiger (1979). Whatever the meaning given to factor scores, the inability to uniquely determine them remains. The indeterminacy of factor scores is complicated by another indeterminacy problem common to all factor analytic models, known as the rotation problem. This indeterminacy arises from the existence of an infinity of equivalent structures. Thus, the factor pattern can be transformed (rotated) to different solutions. This problem has been mostly dealt with by imposing certain structural requirements on the solution — usually simple structure. Both indeterminacy problems are part of the broader problem of identifiability

of parameters. A parameter is said to be identifiable if it has the same value in all equivalent structures. Basically, in factor analysis, there are two ways of eliminating the unidentifiability of the parameters. Each way is specific to unrestricted and restricted models.

In the unrestricted common factor model, arbitrary constraints are chosen either for mathematical convenience in the estimation of the parameters or because they reflect "substantive" considerations or for both reasons. The latter situation seems to apply to some longitudinal models seen in the previous section. For example, in the Corballis-Traub approach the factor scores on different factors are constrained to be uncorrelated within as well as between occasions, whereas the factor scores on the same factors are correlated. Such a selective orthogonality is not only mathematically convenient, it is also a reasonable assumption stemming from the point of view that factors are potential tests.

In the case of restricted models (see Mulaik, 1975, for a review of this topic), the researcher makes a hypothesis related to "the latent parameters present among variables in a domain and will carefully select his variables for factor analysis so as to reveal the presence of the latent parameters as clearly as possible" [Mulaik, 1972, p. 362]. Typically such an approach involves specifying constants or equalities among the elements of the parameters to be estimated. The remaining unspecified

parameters are then estimated so as to maximize a chosen criterion. The resulting estimation gives the model a best overall fit to the data conditional on the specified elements.

In random restricted factor analytic models, the constraints are imposed mostly on the common and unique factors. Though the covariance (or correlation) matrix of the common factors may be constrained, no constraints can be placed directly on the factor scores in random models. However, this could be achieved in a model that allows for simultaneous estimation of the factor loadings and the factor scores. Factor scores as well as factor loadings are then considered as fixed parameters to be estimated. Such a model could be labeled "fixed" as opposed to "random".

Lawley (1942) considered a simultaneous estimation for the unrestricted models, i.e. a fixed common factor model. Anderson and Rubin (1956) showed that in this model the likelihood function does not have a maximum, hence the parameters cannot be estimated. McDonald (1974c, in press) gave the basic equations and methods for the simultaneous estimation of factor loadings and factor scores. He used both a least-squares estimation and a log-likelihood ratio estimation procedure which overcomes the difficulty pointed out by Anderson and Rubin (1956). In this approach it turns out that the optimum value of either the least-squares or the maximum likelihood fit functions is improper; thus, the factor scores are under-identified. McDonald (1974c, 1974d) pointed out that if constraints were imposed on the model, then simultaneous estimates of factor loadings

and factor scores could be obtained. By their very nature, longitudinal studies offer a proper framework for imposing constraints.

Basically two types of constraints could be placed on longitudinal models. First, we could impose restrictive assumptions on the factors that correspond to a priori considerations, e.g., the invariance of the common factor pattern over occasions. Second, we could constrain the factor scores to follow certain laws considered useful in the description of the development of the abilities investigated. These laws which amount to structural constraints on the factor scores can be specified by means of "time-curves". Let us now turn to the topic of specifying adequate laws describing change in factor scores.

In what follows, we will assume that the interrelationships of the different tests of a battery are already accounted for by underlying factors. We will examine some potentially useful relations between time and the unobservable traits or abilities assumed to underlie an examinee's performance on the battery of tests.

The shape of these relations is generally dependent on the domain investigated, as well as on the time span of the study. In empirical studies of development of intelligence and cognitive processes, different researchers using different tests seem to agree as to the general shape of the growth of intelligence. It increases from birth and it reaches an upper limit, that is, a "person reaches a level of basic mental competence which is not exceeded later" (Munn, 1974). There is disagreement

concerning the age at which this upper limit is attained. Whether the basic mental competence is reached at 15 years of age, at 20 years or at 26 years is open to further investigation. It should be noted that recent studies by Bayley (1970) seem to confirm the latter age. However, the matter that most concerns us here is the nonlinear and asymptotic shape of the time-curve.

The problem of finding an adequate representation of the growth of mental ability is discussed by Wohlwill (1973) and Keats (1978a). Halford and Keats (1978) have proposed that the ability a_{ij} of subject i at time t_j is related to his ultimate ability m_i by the expression

$$a_{ij} = \frac{m_i t_j}{t_j + h_i},$$

where h_i is a growth parameter which indicates the time at which the ability is half of the maximum value m_i . In this case, a_{ij} the individual ability on a factor (the factor score) is determined by two parameters. This approach bears some resemblance to the latent trait model proposed by Rasch (1960, 1966a, 1966b) which allows the assessment of ability at a specific time; this is done through the use of an item characteristic curve or function (see also, Lord, 1977 and Wright, 1977). Latent trait models are mostly unidimensional, i.e., there is only one latent trait that underlies an examinee's test performance. Latent trait models which are multidimensional are not well developed to date (Hambleton and Cook, 1977).

Keats' approach provides an integration of the two dominant ways of describing the development of intelligence. The first one describes it in terms of clearly demarcated stages and the second one in terms of steady increments in intelligence (Halford and Keats, 1978; Keats, 1978b). Longitudinal studies in other areas of interest in the social sciences — in particular in education — such as the development of attitudes or the change caused by a sequence of instructions may or may not require such a nonlinear time-course of individual factor scores.

A researcher interested in a relatively short time interval may find a linear time-course specification for the factor scores useful even if the abilities investigated are known to have a nonlinear time-course. The lack of empirical findings may also lead to assuming a linear time-course as a first and/or sufficient step in the research. In this case as well as in the special case where a treatment precedes each measure on each occasion, an unconstrained time-course may prove adequate.

The model presented in Chapter 3 is a fixed restricted factor analytic model. The factor patterns are allowed to change but inter-occasions and within-occasion constraints of interest could be imposed. The factor scores are allowed to change over time according to either unconstrained, linear or nonlinear (Keats) time-course laws.

CHAPTER 3

A FACTOR ANALYTIC MODEL FOR LONGITUDINAL DATA3.1. The Model

Suppose n tests are administered to N persons on p separate occasions. Consider the following factor equation for any given occasion j

$$(3.1) \quad Y_j = F_j X_j + \underline{m}_j \underline{1}'_N + E_j, \quad j = 1, \dots, p$$

$n \times N \quad n \times r \quad r \times N \quad n \times 1 \quad 1 \times N \quad n \times N$

where Y_j and E_j , $(n \times N)'$, are random matrices, F_j , $(n \times r)$ and X_j , $(r \times N)$, are fixed matrices, \underline{m}_j , $(n \times 1)$ is a vector of means and $\underline{1}_N$, $(N \times 1)$, is a vector of unities.

3.2. Assumptions

The assumptions "within occasions" of the model are

$$(3.2) \quad \mathcal{E}(E_j) = 0$$

$$(3.3) \quad \mathcal{E}\left(\frac{1}{N} E_j E_j'\right) = U_{jj},$$

where U_{jj} , $(n \times n)$, is a diagonal matrix.

Further, for the purpose of the likelihood estimation, it is convenient to assume the following:

(3.4) The N columns of E_j are independently and normally distributed.

It then follows that the covariance matrix of Y_j is U_{jj} as can be seen by the following equation

$$\frac{1}{N} [Y_j - \bar{Y}_j][Y_j - \bar{Y}_j]' = \frac{1}{N} E_j E_j' = U_{jj} .$$

The assumptions "among occasions" of the model are

(3.5) The same factors underlie the n tests on each occasion.

(3.6) The factor pattern matrices F_j are invariant over occasions. Let us denote this by

$$F_j = F , \quad j = 1, \dots, p .$$

(3.7) The residual parts E_j are uncorrelated between as well as within occasions, i.e.,

$$\frac{1}{N} E_j E_k' = U_{jk} ,$$

where U_{jk} , $(n \times n)$, is a diagonal matrix.

3.3. Specification of X_j

Three different specifications of X_j will be considered:

- the case where the time-course is unconstrained;
- the case where the time-course is linear;

- the case where the time-course is a prescribed non-linear function.

3.3.1. Case 1 (Time-course unconstrained).

The basic equation for subject s on factor f on a given occasion j is given by the following scalar expression

$$(3.8) \quad x_{fs}^{(j)} = s_{ff}^{(j)} x_{fs} + \mu_f^{(j)}$$

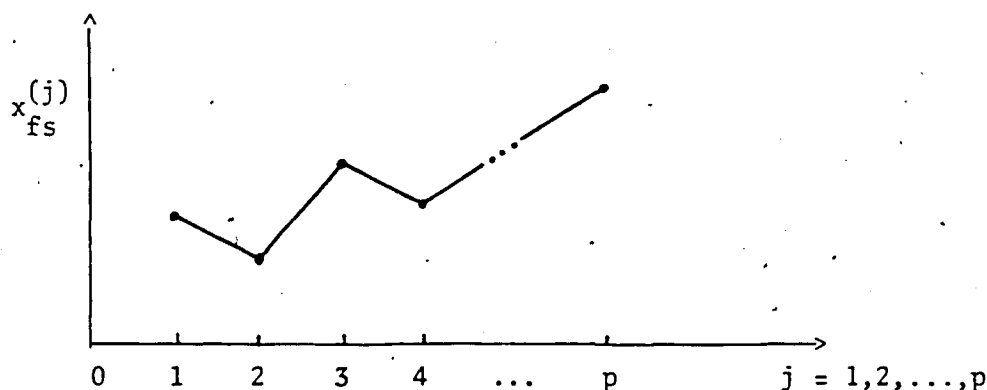


FIGURE 1. The relationship between factor scores and occasions (Time-course unconstrained).

where $s_{ff}^{(j)}$ is a scaling value and $\mu_f^{(j)}$ is the mean of the group. The quantity x_{fs} is the basic score of subject s on factor f .

It seems reasonable to express the factor score of subject s on factor f on occasion j , $x_{fs}^{(j)}$, in terms of the fixed factor score on the same factor only. That is, from the definition of X_j (in matrix notation)

$$(3.8a) \quad X_j = S_j X + \mu_j \mathbf{1}'_N$$

which can be written as

$$(3.8b) \quad \begin{bmatrix} x_{fs}^{(j)} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^r s_{fi}^{(j)} x_{is} + \mu_f^{(j)} \end{bmatrix}$$

where $S_j = \begin{bmatrix} s_{gh}^{(j)} \end{bmatrix}$. By assumption, we set $s_{gh}^{(j)} = 0$ for $g \neq h$.

Thus, S_j is diagonal for any $j = 1, \dots, p$. Equation (3.8b) then yields our original equation (3.8) which is the same as

$$x_{fs} = \frac{x_{fs}^{(j)} - \mu_f^{(j)}}{s_{ff}^{(j)}} \quad , \quad \text{for } \begin{cases} f = 1, \dots, r \\ s = 1, \dots, N \\ j = 1, \dots, p \end{cases}$$

The basic score x_{fs} of subject s on factor f appears to be a "standardized" form of the factor score $x_{fs}^{(j)}$ on any occasion j .

In matrix notation, the specification of X_j on occasion j can be written for one factor ($r=1$) as

$$(3.9) \quad \begin{array}{ccccc} \underline{x}'_j & = & s_j & \underline{x}' & + & \mu_j & \frac{1'}{N} \\ 1 \times N & & & 1 \times N & & & 1 \times N \end{array}$$

constant for all N constant for all N

and for multiple factors ($r > 1$), as

$$(3.10) \quad \begin{array}{ccccc} X_j & = & S_j & X & + & \underline{\mu}_j & \frac{1'}{N} \\ r \times N & & r \times r & r \times N & & r \times 1 & 1 \times N \end{array}$$

diagonal matrix

3.3.2. Case 2 (Linear time-course).

In this case, $x_{fs}^{(j)}$ is a function of time t_j

$$(3.11) \quad x_{fs}^{(j)} = m_{fs} t_j + x_{fs} + \eta_f$$

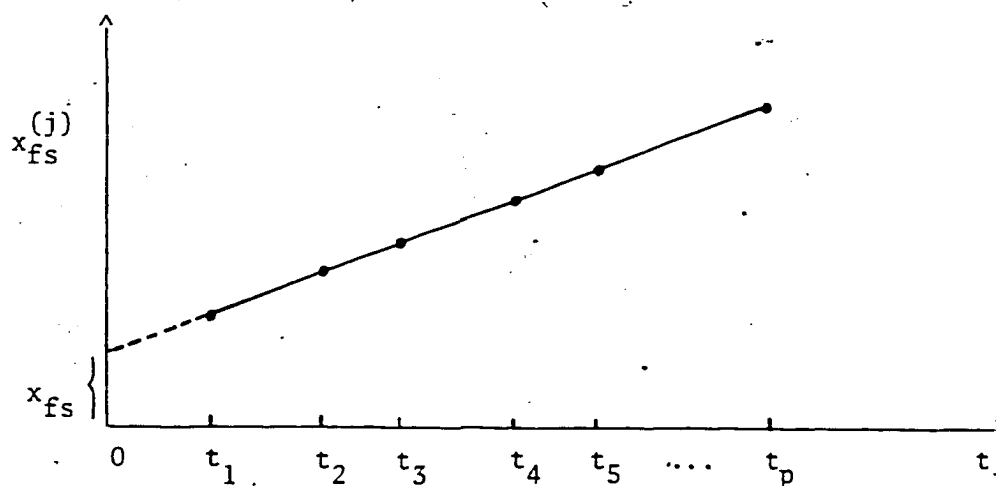


FIGURE 2. The relationship between factor scores and time (Linear time-course).

where m_{fs} is the slope of the straight line, t_j is the time and t_1 can be set to zero (or not), x_{fs} is defined as in Case 1 and η_f is a scaling value.

In matrix notation, the specification of X_j on occasion j can be written for one factor as

$$(3.11a) \quad \begin{array}{ccccccc} \underline{x}_j & = & t_j & \underline{m}' & + & \underline{x}' & + & \eta & \underline{1}'_N \\ 1 \times N & & & 1 \times N & & 1 \times N & & & 1 \times N \end{array}$$

and for multiple factors, as

$$(3.11b) \quad \begin{array}{ccccccc} X_j & = & T_j & M & + & X & + & \underline{\eta} & \underline{1}'_N \\ r \times N & & r \times r & r \times N & & r \times N & & r \times 1 & 1 \times N \end{array}$$

where $T_j = t_j I_r$, M , $(r \times N)$, is the matrix of slopes constant over occasions, X is defined in the same way as in Case 1 and η , $(r \times 1)$, is a scaling vector.

3.3.3. Case 3 (Non-linear time-course for mental growth. Keats).

In this case, we write

$$(3.12) \quad x_{fs}^{(j)} = \frac{x_{fs} t_j}{t_j + h_{fs}}$$

Here, the ability of subject s on occasion j on factor f , $x_{fs}^{(j)}$, is a function of

x_{fs} : the ultimate ability of subject s on factor f

h_{fs} : the time at which development has reached half of the maximum value

t_j : the time on occasion j , where t_1 can be equal to zero or not.

The parameters x_{fs} and h_{fs} are required to be positive, following Keats (1978).

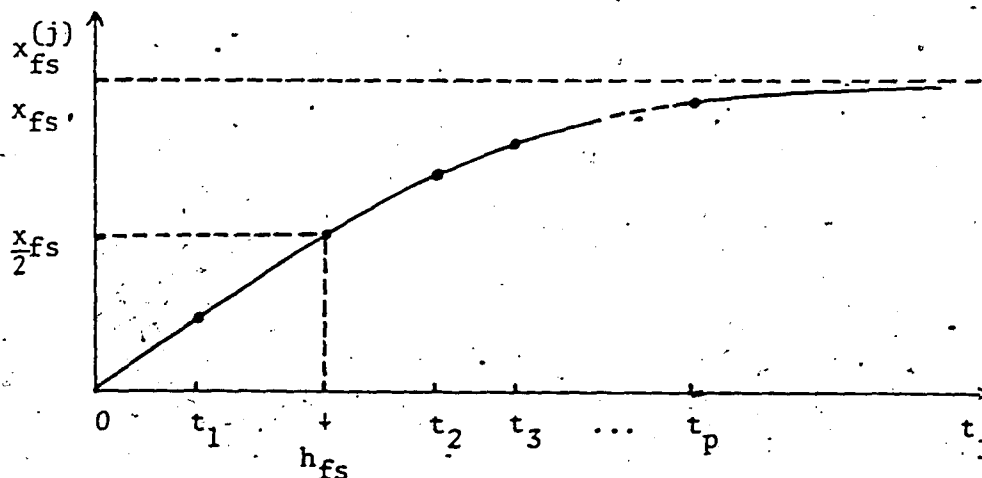


FIGURE 3. The relationship between factor scores and time (Non-linear time-course).

In matrix notation, the specification of X_j on occasion j can be written for one factor as

$$(3.12a) \quad \underline{x}'_j = \left[\frac{x_{11} t_j}{t_j + h_{11}}, \dots, \frac{x_{1N} t_j}{t_j + h_{1N}} \right]$$

$$= \left[x_{11} \left(\frac{1}{1 + h_{11}/t_j} \right), \dots, x_{1N} \left(\frac{1}{1 + h_{1N}/t_j} \right) \right]$$

Setting $k_{1s}^{(j)} = \frac{1}{1 + \frac{h_{1s}}{t_j}}$, $s = 1, \dots, N$, for all j , we obtain

$$(3.12b) \quad \underline{x}'_j = \left[x_{11} k_{11}^{(j)}, \dots, x_{1N} k_{1N}^{(j)} \right]$$

Hence

$$(3.12c) \quad \begin{array}{ccc} \underline{x}'_j & = & \underline{x}' \quad \odot \quad \underline{k}'_j \\ 1 \times N & & 1 \times N \quad 1 \times N \end{array}$$

where $\underline{k}'_j = [k_{11}, \dots, k_{1N}]$, and " \odot " denotes the Hadamard or element-wise product of matrices.

For multiple factors, the specification of X_j on occasion j can be written as

$$(3.12d) \quad \begin{array}{ccc} X_j & = & X \quad \odot \quad K_j \\ r \times N & & r \times N \quad r \times N \end{array}$$

where

$$K_j = \begin{bmatrix} k'_{1j} \\ \vdots \\ k'_{rj} \end{bmatrix} \quad \text{and} \quad \frac{k'_f}{f_j} = [k_{f1}^{(j)}, \dots, k_{fN}^{(j)}]$$

3.4. The general formulation of the model

The general equation of the model over p occasions is given by

$$(3.13) \quad \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix}_{np \times N} = \begin{bmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F \end{bmatrix}_{np \times rp} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}_{rp \times N} + \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_p \end{bmatrix}_{np \times 1} \frac{1'_N}{1 \times N} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_p \end{bmatrix}_{np \times N}$$

Specifying X_j in terms of the fixed factor scores for each of the three cases, we obtain

3.4.1. Case 1.

$$(3.13a) \quad \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix}_{np \times N} = \begin{bmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F \end{bmatrix}_{np \times rp} \left(\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_p \end{bmatrix}_{rp \times r} X + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}_{rp \times 1} \frac{1'_N}{1 \times N} \right) + \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_p \end{bmatrix}_{np \times 1} \frac{1'_N}{1 \times N} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_p \end{bmatrix}_{np \times N}$$

3.4.2. Case 2.

$$(3.13b) \quad \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix} = \begin{bmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F \end{bmatrix} \left(\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_p \end{bmatrix} M + \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix} X + \begin{bmatrix} \underline{n} \\ \underline{n} \\ \vdots \\ \underline{n} \end{bmatrix} \underline{1}'_N \right) + \begin{bmatrix} \underline{m}_1 \\ \underline{m}_2 \\ \vdots \\ \underline{m}_p \end{bmatrix} \underline{1}'_N + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_p \end{bmatrix}$$

$rp \times r \quad r \times N \quad rp \times r \quad r \times N \quad rp \times 1 \quad 1 \times N$

3.4.3. Case 3.

$$(3.13c) \quad \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix} = \begin{bmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F \end{bmatrix} \left(\begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix} X \oplus \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_p \end{bmatrix} \right) + \begin{bmatrix} \underline{m}_1 \\ \underline{m}_2 \\ \vdots \\ \underline{m}_p \end{bmatrix} \underline{1}'_N + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_p \end{bmatrix}$$

$rp \times r \quad r \times N \quad rp \times N$

3.4.4. Supermatrix notation.

Denoting the supermatrices in (3.13) by Y^* , F^* , X^* , \underline{m}^* and E^* , we obtain

$$(3.14) \quad Y^* = F^* X^* + \underline{m}^* \underline{1}'_N + E^*$$

With obvious notation and from (3.13a), (3.13b) and (3.13c), the three specifications of X^* are given by

$$(3.15a) \quad X^* = S^*X + \underline{\mu}^* \underline{1}'_N \quad (\text{Case 1}) ,$$

$$(3.15b) \quad X^* = I^*X + T^*M + \underline{\eta}^* \underline{1}'_N \quad (\text{Case 2}) ,$$

$$(3.15c) \quad X^* = I^*X \odot K^* \quad (\text{Case 3}) .$$

From (3.2), (3.13) and (3.14), we obtain

$$(3.16) \quad \mathcal{E}(Y^*) = F^*X^* + \underline{m}^* \underline{1}'_N ,$$

therefore

$$(3.17) \quad E^* = Y^* - \mathcal{E}(Y^*) .$$

Let W be the covariance matrix of Y^* , that is,

$$(3.18) \quad W = \mathcal{E}\left\{\frac{1}{N} (Y^* - \mathcal{E}(Y^*))(Y^* - \mathcal{E}(Y^*))'\right\} = \mathcal{E}\left\{\frac{1}{N} E^*E^{*'}\right\} .$$

Then

$$(3.19) \quad W = \begin{bmatrix} U_{11} & \dots & U_{1t} \\ \vdots & & \vdots \\ U_{t1} & \dots & U_{tt} \end{bmatrix} ,$$

where U_{jk} is given by (3.7). Since U_{jk} is diagonal, it follows that $U_{jk} = U_{kj}$. The matrix W is symmetric and formed of diagonal blocks.

Let us compare and contrast the random and fixed models.

In the random model, we have

$$(3.20) \quad E\{[\underline{y} - \underline{\mu}][\underline{y} - \underline{\mu}]'\} = C = FF' + U^2,$$

where F , $(n \times r)$, is the common factor pattern and U^2 , $(n \times n)$, is the diagonal matrix of uniquenesses. In the fixed model for longitudinal data, we have

$$(3.21) \quad E\left\{\frac{1}{N} [Y^* - E(Y^*)][Y^* - E(Y^*)]'\right\} = W.$$

In the random model we suppose that $(\underline{y} - \underline{\mu})$ has a multivariate normal distribution with covariance matrix C and zero mean vector. In this fixed model, $(Y^* - F^*X^* - \underline{m}^* \underline{1}'_N) \equiv E^*$ has its N columns independently and normally distributed with covariance matrix W and zero means.

Let Q be defined by

$$(3.22) \quad Q = \frac{1}{N} E^* E^{*'} = \frac{1}{N} (Y^* - F^*X^* - \underline{m}^* \underline{1}'_N)(Y^* - F^*X^* - \underline{m}^* \underline{1}'_N)'$$

This is the conceptual equivalent of the sample covariance matrix in the random model.

3.5. Estimation of the parameters with the Least Squares Method

Using the method of unweighted least squares, we first define

$$(3.23) \quad \psi(F^*, X^*, \underline{m}^*, W) = \frac{1}{2} \text{Tr}(Q - W)^2,$$

where Q is defined by (3.22) and W is patterned as in (3.19). This function is everywhere nonnegative, and becomes zero if and only if $Q = W$.

The function ψ thus measures the error of fit of W to Q . The estimates of the parameters correspond to the minimum of ψ . To set up a method for minimizing ψ we require the derivatives with respect to W, F^*, X^* and \underline{m}^* .

The derivatives are computed using the McDonald-Swaminathan calculus (1973) with further extensions from McDonald (1976). The notation, definitions and theorems that are used in this study are briefly presented in Appendix A.

3.5.1. Derivative of ψ w.r.t. W .

Matrix W is symmetric and formed of diagonal blocks

$$(3.24) \quad W = \begin{bmatrix} U_{11} & \dots & U_{1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ U_{p1} & \dots & U_{pp} \end{bmatrix}$$

Let w_{jk} , the general element of W , be denoted as follows

$$(3.25) \quad w_{jk} = \theta_i \quad \text{if } w_{jk} \text{ is variable,}$$

$$w_{jk} = 0 \quad \text{otherwise.}$$

Matrix W is then denoted by

$$W = \begin{bmatrix} \theta_1 0 & \dots & 0 & \dots & 0 & \dots & \theta_p & 0 & \dots & 0 & \dots & 0 \\ 0 & \theta_{p+1} & \dots & 0 & \dots & 0 & \dots & 0 & \dots & \theta_{2p} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \theta_{(n-1)p+1} & \dots & \theta_{(n-1)p+2} & \dots & 0 & \dots & \theta_{np} & \dots & \theta_{np} \\ \theta_2 0 & \dots & 0 & \dots & 0 & \dots & \theta_{np+p-1} & 0 & \dots & 0 & \dots & 0 \\ 0 & \theta_{p+2} & \dots & 0 & \dots & \theta_{np+p} & \dots & 0 & \dots & \theta_{np+2p-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \theta_{(n-1)p+2} & \dots & \theta_{np+(n-1)(p-1)+1} & \dots & 0 & \dots & \theta_{2np-n} & \dots & \theta_{2np-n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & \dots & \theta_{\frac{np-1+np}{2}} & \dots & \theta_{\frac{np-1+np}{2}} \end{bmatrix}$$

(3.26)

Since this matrix is symmetric, only $\frac{np^2 + np}{2}$ of its elements are mathematically independent and variable. Let these be denoted by

$$(3.27) \quad \underline{\theta} = [\theta_i] \quad , \quad i = 1, \dots, \frac{np^2 + np}{2} .$$

The derivative of ψ w.r.t. $\underline{\theta}$ is given by (see Derivative A.1 in Appendix A):

$$(3.28) \quad \begin{aligned} \frac{\partial \psi}{\partial \underline{\theta}} &= \frac{1}{2} \frac{\partial W}{\partial \underline{\theta}} \frac{\partial (Q - W)}{\partial W} \frac{\partial \text{Tr}(Q - W)^2}{\partial (Q - W)} \\ &= \frac{1}{2} \frac{\partial W}{\partial \underline{\theta}} \left(-\frac{\partial W}{\partial W} \right) [2 \text{Vec}(Q - W)] \\ \underline{\frac{\partial \psi}{\partial \theta}} &= -\frac{\partial W}{\partial \theta} \text{Vec}(Q - W) . \end{aligned}$$

In order to obtain the estimate of $\underline{\theta}$, we solve

$$(3.29) \quad \frac{\partial \psi}{\partial \underline{\theta}} = 0 .$$

It follows that

$$(3.30) \quad \frac{\partial W}{\partial \theta} \text{Vec}(Q - W) = 0 .$$

It follows that a closed form solution is given by

$$(3.31) \quad W = \mathcal{D}_{\text{bloc}}(Q) ,$$

The results displayed in Table 2 show that the estimated vector $\underline{\mu}^*$ changed from the initial vector $\underline{\mu}$ only in the fourth decimal place and beyond. The estimated values of F and S^* are further from the parameter values than are the initial values. At the same time the correlations of factor 1 and factor 2 have changed from $-.05$ at the initial point to $-.22$ (at the LR point) which goes beyond the population value of $-.17$.

However, it is clear from Tables 3 and 4 that the off-diagonal block residuals show a considerable reduction from the initial point to the LS and LR estimated points. Indeed, the LS function goes from $.785$ to $.003$, while the scaled LLR goes from 3.925 to $.013$.

To get further information, the observed matrix Y^* of Example 4.5.1 was reanalysed with the same initial values for F , S^* and $\underline{\mu}^*$ but with different initial values for X . A new initial X was obtained by randomly adding $\pm .01$ to the known population values. Convergence to the criterion occurred in 15 moves for LS, and 21 moves for LR. The results (cf. Table 5) show that the estimated F and S^* are closer to the true parameters than are the estimated values for Example 4.5.1. Furthermore, the residuals of the matrix Q (cf. Table 6 and Table 7) indicate an even better fit than for Example 4.5.1. This fit, $\psi = .000257$ and $\phi = .006664$, is very close to the fit obtained using the exact parameters as starting points: $\psi = .000273$ and $\phi = .006214$. (The covariance matrices for this case are given in Appendix E.) Convergence to the criterion for this latter analysis occurred

in 2 moves for LS and 7 moves for LR.

We have noted that the residuals do indeed become smaller as the algorithm moves from the initial point to the estimated points. The actual final values of the residuals and of the LS loss function (which is directly interpreted as the sum of squares of the off-diagonal block residuals) are all in the order of magnitude that we would expect from the amount of noise added to the data (with standard deviation 0.01, as may be recalled). Let us consider $\sqrt{2\psi/n}$ (where n is the number of off-diagonal block residuals) as a quantity reflecting the absolute magnitude of the typical residual. Then Table 4 yields .0035 and Table E.2 yields .0011, for this quantity. These values are consistent with what inspection suggests. Since the absolute residuals at the solution point are, in order of magnitude, about three times that of the residuals at the population value, and the latter are due to the added noise (of standard deviation .01), it is not implausible to conclude that the estimated points lie within the region about the global minimum that contains all points satisfying the convergence criterion. If this is indeed the case, it is reasonable to conclude further that the discrepancies between the population values of F , S^* , X and the estimated values reflect a tendency of the function to change very slowly about its global minimum. This conclusion is somewhat unexpected, since it was noted above in Section 4.3.3 that in Case 1 the parameters should in general be identified at least in respect of the usual oblique or orthogonal transformations of the classical

(one-occasion) common factor model. The matrices S^* in Tables 2 and 5 are in fact very close to scalar matrices. If the factor loading matrix had been unconstrained, we would therefore expect the function to behave almost as though there were such rotational indeterminacy present. However, since the factor loading matrix has been constrained to have a simple structure, this explanation seems to be ruled out. It nevertheless remains plausible to suppose that we have found points in the region of the global minimum (satisfying the convergence criterion) and that this region is larger than we might wish it to be.

The correlations of the population factor scores with the estimates show very little gain over the correlations with the initial values. It will be seen below in the contrasting example of Case 3, that this is quite possibly because the initial correlations are very high.

We note that during minimization $\underline{\mu}^*$ in this case changes only in the fourth decimal place and beyond. Similarly, in the further cases below $\underline{\mu}^*$, and also $\underline{\eta}$ in Case 2, never change up to the second decimal place. We remark at this point that the derivatives of the loss functions with respect to these parameters vanish everywhere, when \underline{m}^* is appropriately chosen. Hence we might expect very little movement in conjugate directions, and certainly no movement at all in the direction of steepest descent. This fact seems to correspond to a possible joint indeterminacy of the factor scores and mean vectors associated with the model. This is a matter for further investigation in future research.

Example 4.5.2 (Case 1)

In this example the parameters of Case 1 are patterned in the following way

$$F = \begin{bmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \\ \alpha_3 & 0 \\ 0 & \alpha_4 \\ 0 & \alpha_5 \\ 0 & \alpha_6 \end{bmatrix}, \quad S^* = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \\ \beta_3 & 0 \\ 0 & \beta_4 \\ \beta_5 & 0 \\ 0 & \beta_6 \\ \beta_7 & 0 \\ 0 & \beta_8 \end{bmatrix}, \quad \underline{\mu}^* = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_1 \\ \eta_2 \\ \eta_1 \\ \eta_2 \\ \eta_1 \\ \eta_2 \end{bmatrix}.$$

Let us note that $\underline{\mu}^*$ is patterned differently than in the preceding example. The total number of independent parameters is 136. The numerical results are given in Table 8, and the covariance matrices in Table 9 and Table 10.

Number of iterations to convergence: 22 for LS, and 36 for LR.

The estimated parameters displayed in Table 8 show similar departures from the parameter values to those observed in Example 4.5.1. However, as before, the matrices Q given in Table 9 and Table 10 show a satisfactory reduction of the off-diagonal block residuals to an order of magnitude consistent with that of the noise added to the data.

As in the preceding example, a reanalysis of the data using the same initial values for F , S^* and $\underline{\mu}^*$ but with different initial values for X was performed. A new initial X was obtained by randomly adding ± 0.01 to the parameter values.

The results (cf. Appendix E) are closer to the true parameters than are the estimated values for Example 4.5.2. The matrices Q obtained from this reanalysis show a better fit than for Example 4.5.2. This fit, $\psi = .000260$ and $\phi = .006047$, is very close to the fit obtained using the exact parameters as starting points: $\psi = .000280$ and $\phi = .005986$. (The Q matrices for this latter case are given in Appendix E.)

Again it is not implausible to suppose that we are finding points in the region about the global minimum that satisfy the convergence criterion.

Example 4.5.3 (Case 2)

In Case 2, only matrix F and vector $\underline{\eta}$ require a pattern. In this example, F and $\underline{\eta}$ are patterned in the following way

$$F = \begin{bmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \\ \alpha_3 & 0 \\ 0 & \alpha_4 \\ 0 & \alpha_5 \\ 0 & \alpha_6 \end{bmatrix}, \quad \underline{\eta} = \begin{bmatrix} \alpha_7 \\ \alpha_8 \end{bmatrix}.$$

The times are set at $t = 1, 2, 3, 4$. The total number of independent parameters is 248. The numerical results and the corresponding covariance matrices are given in the following tables.

Number of iterations to convergence: 112 for LS, and 197 for LR.

The estimated parameters displayed in Table 11 show a similar departure from the parameter values to the one observed in the previous examples. The data were reanalysed using the same initial values for F and η ; new initial values for X and M were obtained by randomly adding $\pm .01$ to the parameter values.

Again, from the magnitudes of the residuals (cf. Tables 12, 13 and Appendix E), we appear to be finding points in the region around the global minimum.

TABLE 11 (continued)

Population parameters	Initial values	Estimated parameters (LS solution)	Estimated parameters (LR solution)
Correlation of factor 1 with factor 2 in matrix X			
.092	.082	.068	.040

Correlations summarizing behaviour of factor scores in matrix X

(Lower triangle: factor 1 ; upper triangle: factor 2)

	Pop.	In.	LS	LR
Population		.796	.797	.800
Initial	.883		1.000	1.000
LS	.893	.981		1.000
LR	.904	.975	.996	

A graphical representation of individual Keats-curves for five randomly drawn subjects and of the overall arithmetic mean curve (over 60 subjects) is given in Figure 4 and Figure 5.

4.6. Real data - Three Examples

The data used in this section originate from a data bank of a long term study of program and architecture in Ontario schools by Weiss and Traub (1978). As part of a questionnaire, 204 teachers responded to 45 items on a five point scale of agreement during the school years 1973-1974, 1974-1975 and 1975-1976.

The data were factor analysed and reduced to 100 subjects and 6 variables with a stable simple structure 2 factor solution for each year. Hence, in the notation of our model, $n = 6$, $N = 100$, $r = 2$ and $p = 3$. More details regarding the reduction of the original data set as well as a list of the original 45 questions are found in Appendix F.

The following examples are based on the resulting reduced matrix of observed data Y^* , (18×100) . All three specifications of the factor scores were used. The initial values were computed using JFACTOR and the appropriate program INPT; they are listed in Appendix G. The best step bounds (cf. Section 5.3.2, equation 5.17) were generally found to be .5 and .05 respectively for the least squares solutions and for the likelihood ratio solutions. The criterion of convergence was set at $\epsilon = 10^{-2}$. This is due to an excessively large amount of time needed for

for a total number of independent parameters equal to 218. The numerical results for the two sets of patterns are given in the preceding tables. The initial values are listed in Appendix G.

TABLE 20

Summary of results for Example 4.6.1

	Set 1		Set 2	
	LS	LR	LS	LR
Number of iterations to convergence ($\epsilon=10^{-2}$)	429	441	344	385
Initial value of loss function	24.262	2.901	28.939	3.483
Final value of loss function	5.160	1.663	5.470	1.692

We also analysed Set 1 with $\epsilon = 10^{-3}$. Convergence was attained after 1036 iterations and 3800 CP seconds of execution time.

TABLE 24
 Summary of results
 for Example 4.6.2

	Set 1		Set 2	
	LS	LR	LS	LR
Number of iterations to convergence ($\epsilon=10^{-2}$)	220	248	367	403
Initial value of loss function	3.705	1.763	1661.357	64.741
Final value of loss function	3.592	1.126	3.666	1.079

Example 4.6.3. (Case 3)

In this example the data have been analysed according to the specifications of Case 3. The times are set at $t = 1, 2, 3$. The parameters of F are specified in the following way

$$F = \begin{bmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \\ \alpha_3 & 0 \\ 0 & \alpha_4 \\ 0 & \alpha_5 \\ 0 & \alpha_6 \end{bmatrix}$$

The number of independent parameters is equal to 406. The algorithm failed to converge at the criterion of convergence $\epsilon = 10^{-2}$ for values of the step bounds varying from 10^{-5} to 10. Though no convergence was attained, the loss functions ψ and ϕ decreased in value from the initial to the final point. The lowest final values of ψ and ϕ were attained for the values of the step bounds equal to 9 and .09 respectively for the least squares and the likelihood ratio solutions. As shown by the following table, the corresponding matrices Q showed an overall reduction of the nondiagonal elements of the diagonal blocks. The initial values are listed in Appendix G.

The failure of the algorithm to converge in Case 3 could be due to a number of single reasons or, more probably, to a combination of reasons. Among the latter, let us mention the basic slowness of the minimizing algorithm in the vicinity of a minimum combined with a starting point possibly too far away from a minimum.

An obvious difficulty with a real data set is that we have no standard of comparison whereby we may decide whether the model is giving a reasonable account of the data. Here, the residuals are what might be considered "large". It might be that we have the best fit obtainable, but the best fit is in fact not good by the standards of conventional factor analysis. Another possibility is that we have found a local minimum.

A possible basis for comparison of the model with more established methods of analysis exists at least in Case 1. There is a reasonably obvious model for longitudinal factor analysis with random factor scores that can be regarded as closely analogous to Case 1. Consider the model for an $(np \times np)$ covariance matrix,

$$(4.32) \quad C = F^* S^* P S^{*'} F^{*'} + W$$

where F^* , S^* , and W are defined as in (3.14), (3.15a), and (3.19), and P is an $(r \times r)$ correlation matrix of random factor scores. We might expect that if the $(np \times N)$ matrix Y^* fits Case 1 of the model to some reasonable approximation, then it should also fit the covariance structure given by (4.32) to some reasonable approximation.

A program COSAN has been written by C. Fraser to embody a general model for the analysis of covariance structures given by McDonald (1978). This program was applied to fit the model (4.32) to our real data set by least squares and by maximum likelihood:

a) with uncorrelated factors, b) with correlated factors.

The LS estimated matrices F^* , and S^* , for uncorrelated factors

($P = I_2$) are given in Appendix H, together with the resulting residual matrix. It is clear by inspection that the random factors model gives a worse fit than Case 1 of the fixed factors model. The other three analyses, LS for correlated factors, and ML for both uncorrelated and correlated factors, gave very similar results, and are not reported. We note, however, that the ML solution, by well known theory, yields an asymptotic chi-square test which is highly significant. (That is, we obtain a chi-square of 208.802 on 117 df, $p < .0000$).

The LS function in COSAN is strictly comparable to the LS function fitted in the present work. Table 26 summarizes the values of the loss functions obtained in the fixed factor score analyses of the real data sets, with the LS function from the analysis by (4.32) as a standard of comparison.

TABLE 26

Loss function values for the real data set

		LS		LR	
		Initial	Final	Initial	Final
Case 1	Set 1	24.262	5.160	2.901	1.663
Case 1	Set 2	28.939	5.470	3.483	1.692
Case 2	Set 1	3.700	3.592	1.763	1.126
Case 2	Set 2	1661.357	3.666	64.741	1.079
Case 3		198.402	28.246	4.789	3.008
Random model		n/a	11.834	n/a	2.088

From Table 26, it is clear that Case 2 gave best fit to the data and that all the fixed model analyses except that of Case 3 gave better fit to the data than did the random model (Case 1).

Let us now try to interpret the real data. We will base our interpretation on the results obtained in Case 2 (Example 4.6.2), as it is the case in which the model best fits the data.

The estimated parameters of F (cf. Table 21) confirm the existence of a clear simple structure that is stable over time. The first factor is defined by variables 1, 2, and 3, and the second factor by variables 4, 5 and 6. Based on the information given in Appendix F, we propose the following definitions of the factors.

Factor 1 defines the overall satisfaction of the teachers with the teaching profession. Factor 2 defines the openness of the teachers towards educational experimentation and innovation.


We can now determine individual attitudes as well as the overall attitude of the group. We will do this by using the estimated parameters for Set 2 since it imposes a simple structure on the data and it has the best fit (cf. Table 24). Since both the least squares and the likelihood solutions are similar we will use only the latter.

First, let us recall that in Case 2, the elements of X (the matrix of basic factor scores) are standardized. Hence, the arithmetic mean over the whole group of 100 teachers is zero. Therefore, the

individual basic score x_{fs} of subject s on factor f is readily related to the group. To simplify the following individual interpretations we will label a subject's attitude on a factor as originally "high" or "low" if a positive or negative basic factor score was obtained.

The elements of M , the matrix of slopes, are not standardized. The arithmetic means could therefore be interpreted as an indication of the overall rate of change of the attitudes with respect to each factor. In this case, the means are practically null (.034 for factor 1 and -.005 for factor 2), thus hinting to a stable rate of change over time.

We will illustrate the interpretation of the individual factor scores over time by selecting the scores of 3 subjects whose attitudes are typically different. A graphical representation is given in Figure 6.

Subject 1 shows a decrease of the originally high overall satisfaction with teaching coupled with a similar decrease on the "openness" factor. Subject 72 shows an increase of the originally low overall satisfaction with teaching to a final high satisfaction. This subject also shows an increase on the originally high "openness" factor. Finally, the overall satisfaction with teaching of subject 37 is increasing from an originally high degree, whereas his (her) openness towards educational experimentation and innovation is decreasing from a high degree of satisfaction to a low one. 

CHAPTER 5

DISCUSSION, SUMMARY AND CONCLUSIONS

A review of the main existing models for longitudinal data in the social sciences was presented in Chapter 2. The need for a model allowing an investigation of the latent structure of the variables as well as giving information on individual trends was established. It was shown that designing a factor analytic model which allows the factor patterns to change and whose factor scores follow certain laws is a feasible strategy of analysis. This approach raised the problem of the identifiability of the parameters and the problem of specifying adequate laws of change. The need to reduce the indeterminacy of factor scores, combined with the specification of laws of change for the factor scores, led us to consider the factor scores as well as the factor loadings as fixed parameters to be estimated.

Basically the model presented in Chapter 3 is developed using the notion of McDonald's (1974c, 1974d, in press) simultaneous estimation of factor scores and loadings. Three sets of specifications of the factor scores corresponding to three laws depicting individual change were presented. The first law is defined as an unconstrained time-course, the other two laws as linear and nonlinear time-courses. In this study, two estimation procedures were developed: a least squares method and a likelihood ratio method. The latter is based on a log-likelihood ratio which circumvents the problem outlined by Anderson and Rubin (1956).

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